

# THE TOPOLOGICAL TRANSVERSAL TVERBERG THEOREM PLUS CONSTRAINTS

PAVLE V. M. BLAGOJEVIĆ, ALEKSANDRA S. DIMITRIJEVIĆ BLAGOJEVIĆ, AND GÜNTER M. ZIEGLER

**ABSTRACT.** In this paper we use the strength of the constraint method in combination with a generalized Borsuk–Ulam type theorem and a cohomological intersection lemma to show how one can obtain many new topological transversal theorems of Tverberg type. In particular, we derive a topological generalized transversal Van Kampen–Flores theorem and a topological transversal weak colored Tverberg theorem.

## 1. INTRODUCTION

At the Symposium on Combinatorics and Geometry in Stockholm 1989, Helge Tverberg formulated the following conjecture that in a special case coincides with his famous 1966 result [14, Thm. 1].

**Conjecture 1.1** (The transversal Tverberg conjecture). *Let*

- $m$  and  $d$  be integers with  $0 \leq m \leq d - 1$ ,
- $r_0, \dots, r_m \geq 1$  be integers, and
- $N_0 = (r_0 - 1)(d + 1 - m), \dots, N_m = (r_m - 1)(d + 1 - m)$ .

*Then for every collection of sets  $X_0, \dots, X_m \subset \mathbb{R}^d$  with  $|X_0| = N_0 + 1, \dots, |X_m| = N_m + 1$ , there exist an  $m$ -dimensional affine subspace  $L$  of  $\mathbb{R}^d$  and  $r_\ell$  pairwise disjoint subsets  $X_\ell^1, \dots, X_\ell^{r_\ell}$  of  $X_\ell$ , for  $0 \leq \ell \leq m$ , such that*

$$\text{conv}(X_0^1) \cap L \neq \emptyset, \dots, \text{conv}(X_0^{r_0}) \cap L \neq \emptyset, \dots, \text{conv}(X_m^1) \cap L \neq \emptyset, \dots, \text{conv}(X_m^{r_m}) \cap L \neq \emptyset.$$

For  $m = 0$  this conjecture is Tverberg’s well-known theorem. Tverberg and Vrećica published the full conjecture in 1993 [15]. They proved that it also holds for  $m = d - 1$  [15, Prop. 3]. For  $m = 1$  and arbitrary  $d$  they verified the conjecture only in the following three cases:  $r_0 = 1, r_1 = 1$ , and  $r_0 = r_1 = 2$  [15, Prop. 1].

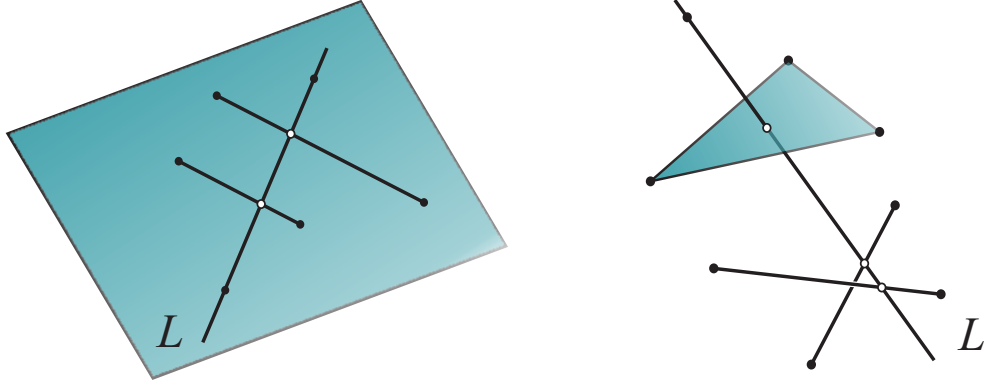


FIGURE 1. The transversal Tverberg conjecture for  $m = 1, r_0 = r_1 = 2$  and  $d = 2$  or  $d = 3$ .

The classical Tverberg theorem from 1966 was extended to a topological setting by Bárány, Shlosman, and Szűcs [2] in 1981. Similarly, it is natural to consider the following extension of the transversal Tverberg conjecture.

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**Conjecture 1.2** (The topological transversal Tverberg conjecture). *Let*

- $m$  and  $d$  be integers with  $0 \leq m \leq d - 1$ ,
- $r_0, \dots, r_m \geq 1$  be integers, and
- $N_0 = (r_0 - 1)(d + 1 - m), \dots, N_m = (r_m - 1)(d + 1 - m)$ .

*Then for every collection of continuous maps  $f_0: \Delta_{N_0} \rightarrow \mathbb{R}^d, \dots, f_m: \Delta_{N_m} \rightarrow \mathbb{R}^d$  there exist an  $m$ -dimensional affine subspace  $L$  of  $\mathbb{R}^d$  and  $r_\ell$  pairwise disjoint faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  such that*

$$f_0(\sigma_1^0) \cap L \neq \emptyset, \dots, f_0(\sigma_{r_0}^0) \cap L \neq \emptyset, \quad \dots, \quad f_m(\sigma_1^m) \cap L \neq \emptyset, \dots, f_m(\sigma_{r_m}^m) \cap L \neq \emptyset.$$

In 1999 using advanced methods of algebraic topology Živaljević [19, Thm. 4.8] proved this conjecture for  $d$  and  $m$  odd integers and  $r_0 = \dots = r_m$  being an odd prime. The topological transversal Tverberg conjecture was settled for  $r_0 = \dots = r_m = 2$  by Vrećica [17, Thm. 2.2] in 2003. In 2007 Karasev [10, Thm. 1] established the topological transversal Tverberg conjecture in the cases when integers  $r_0, \dots, r_m$  are, not necessarily equal, powers of the same prime  $p$  and the product  $p(d - m)$  is even.

In the same paper Karasev [10] proved a colored topological transversal Tverberg's theorem [10, Thm. 5], which for  $m = 0$  coincides with the colored Tverberg theorem of Živaljević and Vrećica [20, Thm. pp.1] and colored Tverberg theorem of type B of Živaljević and Vrećica [18, Thm. 4]. In 2011 Blagojević, Matschke and Ziegler gave yet another colored topological transversal Tverberg theorem [5, Thm. 1.3] that in the case  $m = 0$  coincides with their optimal colored Tverberg theorem [6, Thm. 2.1].

The existence of counterexamples to the topological Tverberg conjecture for non-primepowers, obtained by Frick [9] [3] based on the remarkable work of Mabillard and Wagner [11] [12], in particular invalidates Conjecture 1.2 in the case when  $m = 0$  and  $r_0$  is not a prime power.

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## 2. STATEMENT OF THE MAIN RESULTS

In 2014 Blagojević, Frick and Ziegler [4] introduced the “constraint method,” by which the topological Tverberg theorem implies almost all its extensions, which had previously been obtained as substantial independent results, such as the “Colored Tverberg Theorem” of Živaljević and Vrećica [20] and the “Generalized Van Kampen–Flores Theorem” of Sarkaria [13] and Volovikov [16]. Thus the constraint method reproduced basically all Tverberg type theorems obtained during more than three decades with a single elementary idea. Moreover, the constraint method in combination with the work of Mabillard and Wagner on the “ $r$ -fold Whitney trick” [11] [12] yields counterexamples to the topological Tverberg theorem for non-prime powers, as demonstrated by Frick [9] [3].

In this paper we use the constraint method in combination with a generalized Borsuk–Ulam type theorem and a cohomological intersection lemma to show how one can obtain many new topological transversal theorems of Tverberg type. We prove in detail a new generalized transversal van Kampen–Flores theorem and a new topological transversal weak colored Tverberg theorem.

**Theorem 2.1** (The topological generalized transversal Van Kampen–Flores theorem). *Let*

- $m$  and  $d$  be integers with  $0 \leq m \leq d - 1$ ,
- $r_0 = p^{e_0}, \dots, r_m = p^{e_m}$  be powers of the prime  $p$ , where  $e_0, \dots, e_m \geq 0$  are integers,
- $N_0 = (r_0 - 1)(d + 2 - m), \dots, N_m = (r_m - 1)(d + 2 - m)$ ,
- $k_0 = \lceil \frac{r_0 - 1}{r_0} d \rceil, \dots, k_m = \lceil \frac{r_m - 1}{r_m} d \rceil$ , and
- $p(d - m)$  be even, or  $m = 0$ .

*Then for every collection of continuous maps  $f_0: \Delta_{N_0} \rightarrow \mathbb{R}^d, \dots, f_m: \Delta_{N_m} \rightarrow \mathbb{R}^d$  there exist an  $m$ -dimensional affine subspace  $L$  in  $\mathbb{R}^d$  and  $r_\ell$  pairwise disjoint faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  in the  $k_\ell$ -skeleton of  $\Delta_{N_\ell}$ , for  $0 \leq \ell \leq m$ , such that*

$$f_0(\sigma_1^0) \cap L \neq \emptyset, \dots, f_0(\sigma_{r_0}^0) \cap L \neq \emptyset, \quad \dots, \quad f_m(\sigma_1^m) \cap L \neq \emptyset, \dots, f_m(\sigma_{r_m}^m) \cap L \neq \emptyset.$$

The special case  $r_0 = \dots = r_m = 2$  of the previous theorem is due to Karasev [10, Cor. 4].

In order to state the next result we recall the notion of a rainbow face. Suppose that the vertices of the simplex  $\Delta_N$  are partitioned into color classes  $\text{vert}(\Delta_N) = C_0 \sqcup \dots \sqcup C_k$ . The subcomplex  $R_N := C_0 * \dots * C_k \subseteq \Delta_N$  is called the *rainbow complex*, that is, the subcomplex of all faces that have at most one vertex of each color class  $C_0, \dots, C_k$ . Faces of  $R_N$  are called *rainbow faces*.

**Theorem 2.2** (The topological transversal weak colored Tverberg theorem). *Let*

- $m$  and  $d$  be integers with  $0 \leq m \leq d-1$ ,
- $r_0 = p^{e_0}, \dots, r_m = p^{e_m}$  be powers of the prime  $p$ , where  $e_0, \dots, e_m \geq 0$  are integers,
- $N_0 = (r_0 - 1)(2d + 2 - m), \dots, N_m = (r_m - 1)(2d + 2 - m)$ ,
- the vertices of the simplex  $\Delta_{N_\ell}$ , for every  $0 \leq m \leq d$ , be colored by  $d+1$  colors, where each color class has cardinality at most  $2r_\ell - 1$ ,
- $p(d-m)$  be even, or  $m = 0$ .

Then for every collection of continuous maps  $f_0: \Delta_{N_0} \rightarrow \mathbb{R}^d, \dots, f_m: \Delta_{N_m} \rightarrow \mathbb{R}^d$  there exist an  $m$ -dimensional affine subspace  $L$  of  $\mathbb{R}^d$  and  $r_\ell$  pairwise disjoint rainbow faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  in  $\Delta_{N_\ell}$ , for  $0 \leq \ell \leq m$ , such that

$$f_0(\sigma_1^0) \cap L \neq \emptyset, \dots, f_0(\sigma_{r_0}^0) \cap L \neq \emptyset, \quad \dots, \quad f_m(\sigma_1^m) \cap L \neq \emptyset, \dots, f_m(\sigma_{r_m}^m) \cap L \neq \emptyset.$$

The proofs of Theorems 2.1 and 2.2 are almost identical; see Sections 4.1 and 4.2. The only difference occurs in the definition of the bundles  $\xi_\ell$  and  $\tau_\ell$ , in (1) and (2), and the bundle maps  $\Phi_\ell$  for  $0 \leq \ell \leq m$ ; see Sections 4.1.2 and 4.2.2. Using the same proof technique as for these theorems and modifying the bundles  $\xi_\ell$  and bundle maps  $\Phi_\ell$  using recipes from [4, Lem. 4.2], one can also derive, for example, a topological transversal colored Tverberg theorem of type B, a topological transversal Tverberg theorem with equal barycentric coordinates, or mixtures of those. The most general transversal Tverberg theorem that is produced by the constraint method can be formulated using the concept of “Tverberg unavoidable subcomplexes” [4, Def. 4.1], as follows.

Let  $r \geq 2$ ,  $d \geq 1$  and  $N \geq r-1$  be integers, and let  $f: \Delta_N \rightarrow \mathbb{R}^d$  be a continuous map with at least one Tverberg  $r$ -partition, that is, a collection of  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ . A subcomplex  $\Sigma$  of the simplex  $\Delta_N$  is *Tverberg unavoidable with respect to  $f$*  if for every Tverberg partition  $\{\sigma_1, \dots, \sigma_r\}$  of  $f$  there exists at least one face  $\sigma_i$  that lies in the subcomplex  $\Sigma$ .

**Theorem 2.3** (A constraint topological transversal Tverberg theorem). *Let*

- $m$  and  $d$  be integers with  $0 \leq m \leq d-1$ ,
- $c_1, \dots, c_m \geq 0$  be integer,
- $r_0 = p^{e_0}, \dots, r_m = p^{e_m}$  be powers of the prime  $p$ , where  $e_0, \dots, e_m \geq 0$  are integers,
- $N_0 = (r_0 - 1)(d + 1 + c_1 - m), \dots, N_t = (r_t - 1)(d + 1 + c_m - m)$ ,
- $\Sigma_{i,j}$  be a Tverberg unavoidable subcomplex of the simplex  $\Delta_{N_i}$  with respect to any continuous map  $\Delta_{N_i} \rightarrow \mathbb{R}^d$  for  $1 \leq i \leq m$  and  $0 \leq j \leq c_i$ , assuming that  $\Sigma_{i,0} = \Delta_{N_i}$ , and
- $p(d-m)$  be even, or  $m = 0$ .

Then for every collection of continuous maps  $f_0: \Delta_{N_0} \rightarrow \mathbb{R}^d, \dots, f_m: \Delta_{N_m} \rightarrow \mathbb{R}^d$  there exist an  $m$ -dimensional affine subspace  $L$  of  $\mathbb{R}^d$  and  $r_\ell$  pairwise disjoint faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  that belong to the subcomplex  $\Sigma_{\ell,0} \cap \dots \cap \Sigma_{\ell,c_\ell}$ , for  $0 \leq \ell \leq m$ , such that

$$f_0(\sigma_1^0) \cap L \neq \emptyset, \dots, f_0(\sigma_{r_0}^0) \cap L \neq \emptyset, \quad \dots, \quad f_m(\sigma_1^m) \cap L \neq \emptyset, \dots, f_m(\sigma_{r_m}^m) \cap L \neq \emptyset.$$

### 3. A GENERALIZED BORSUK–ULAM TYPE THEOREM AND TWO LEMMAS

In this section, we present the topological methods, developed in [5] and [10], that we will use in the proofs of Theorems 2.1 and 2.2. In particular, we will review and slightly modify a generalized Borsuk–Ulam type theorem [5, Thm. 4.1], give an intersection lemma [5, Lem. 4.3] and recall the Euler class computation of Dol’nikov [7, Lem. p. 2], Živaljević [19, Prop. 4.9], and Karasev [10, Lem. 8].

**3.1. Fadell–Husseini index.** In 1988 Fadell and Husseini [8] introduced an ideal-valued index theory for the category of  $G$ -space, or more general for the category of  $G$ -equivariant maps to a fixed space with a trivial  $G$ -action. We give an overview of the index theory adjusted to the needs of this paper.

Let  $G$  be a finite group, let  $R$  be a commutative ring with unit, and let  $B$  be a space with the trivial  $G$  action. For a  $G$ -equivariant map  $p: X \rightarrow B$  and a ring  $R$ , the *Fadell–Husseini index* of  $p$  is defined to be the kernel ideal of the map in the equivariant Čech cohomology with coefficients in the the ring  $R$  induced by  $p$ :

$$\begin{aligned} \text{Index}_G^B(p; R) &= \ker(p^*: H^*(EG \times_G B; R) \rightarrow H^*(EG \times_G X; R)) \\ &= \ker(p^*: H_G^*(B; R) \rightarrow H_G^*(X; R)). \end{aligned}$$

The equivariant cohomology of a  $G$ -space  $X$  is assumed to be the Čech cohomology of the Borel construction  $EG \times_G X$  associated to the space  $X$ .

The basic properties of the index are:

- *Monotonicity:* If  $p: X \rightarrow B$  and  $q: Y \rightarrow B$  are  $G$ -equivariant maps, and  $f: X \rightarrow Y$  is a  $G$ -equivariant map such that  $p = q \circ f$ , then

$$\text{Index}_G^B(p; R) \supseteq \text{Index}_G^B(q; R).$$

- *Additivity:* If  $(X_1 \cup X_2, X_1, X_2)$  is an excisive triple of  $G$ -spaces and  $p: X_1 \cup X_2 \rightarrow B$  is a  $G$ -equivariant map, then

$$\text{Index}_G^B(p|_{X_1}; R) \cdot \text{Index}_G^B(p|_{X_2}; R) \subseteq \text{Index}_G^B(p; R).$$

- *General Borsuk–Ulam–Bourgin–Yang theorem:* Let  $p: X \rightarrow B$  and  $q: Y \rightarrow B$  be  $G$ -equivariant maps, and let  $f: X \rightarrow Y$  be a  $G$ -equivariant map such that  $p = q \circ f$ . If  $Z \subseteq Y$  then

$$\text{Index}_G^B(p|_{f^{-1}(Z)}; R) \cdot \text{Index}_G^B(q|_{Y \setminus Z}; R) \subseteq \text{Index}_G^B(p; R).$$

In the case when  $B$  is a point and  $p: X \rightarrow B$  is a  $G$ -equivariant map we simplify notation and write  $\text{Index}_G^B(p; R) = \text{Index}_G^{\text{pt}}(X; R)$ . With this, the next property of the index can be formulated as follows.

- If  $X$  is a  $G$ -space and  $p: B \times X \rightarrow B$  is the projection on the first factor, then

$$\text{Index}_G^B(p; R) = \text{Index}_G^{\text{pt}}(X; R) \otimes H^*(B; R).$$

**3.2. A generalized Borsuk–Ulam type theorem.** The cohomology of the elementary abelian groups  $(\mathbb{Z}/p)^e$ , where  $p$  is a prime and  $e \geq 1$  is an integer, is given by

$$\begin{aligned} H^*((\mathbb{Z}/2)^e; \mathbb{F}_2) &= \mathbb{F}_2[t_1, \dots, t_e], & \deg t_j &= 1 \\ H^*((\mathbb{Z}/p)^e; \mathbb{F}_p) &= \mathbb{F}_p[t_1, \dots, t_e] \otimes \Lambda[u_1, \dots, u_e], & \deg t_j &= 2, \deg u_i = 1 \text{ for } p \text{ odd.} \end{aligned}$$

The following theorem and its proof is just a slight modification of [5, Thm. 4.1].

**Theorem 3.1** (Borsuk–Ulam type theorem). *Let*

- $G = (\mathbb{Z}/p)^e$  be an elementary abelian group where  $p$  is a prime and  $e \geq 1$ ,
- $B$  be a connected space with the trivial  $G$ -action,
- $q: E \rightarrow B$  be a  $G$ -equivariant vector bundle where all fibers carry the same  $G$ -representation,
- $q|_{E^G}: E^G \rightarrow B$  be the fixed-point subbundle of the vector bundle  $q: E \rightarrow B$ ,
- $q|_C: C \rightarrow B$  be its  $G$ -invariant orthogonal complement subbundle ( $E = C \oplus E^G$ ),
- $F$  be the fiber of the vector bundle  $q_C: C \rightarrow B$  over the point  $b \in B$ ,
- $0 \neq \alpha \in H^*(G; \mathbb{F}_p)$  be the Euler class of the vector bundle  $F \rightarrow EG \times_G F \rightarrow BG$ , and
- $K$  be a  $G$ -CW-complex such that  $\alpha \notin \text{Index}_G^{\text{pt}}(K; \mathbb{F}_p)$ .

Assume that

- $\pi_1(B)$  acts trivially on  $H^*(F; \mathbb{F}_p)$ , and
- we are given a  $G$ -equivariant map  $\Phi: B \times K \rightarrow E$  such that the following diagram commutes

$$\begin{array}{ccc} B \times K & \xrightarrow{\Phi} & E = C \oplus E^G \\ & \searrow q_1 & \swarrow q \\ & B & \end{array}$$

where  $q_1: B \times K \rightarrow B$  is the projection on the first coordinate.

Then for

$$S := \Phi^{-1}(E^G) \quad \text{and} \quad T := \Phi(S) = \text{im}(\Phi) \cap E^G$$

the following maps, induced by the projections  $q_1$  and  $q$ , are injective:

$$(q_1|_S)^*: H^*(B; \mathbb{F}_p) \rightarrow H_G^*(S; \mathbb{F}_p) \quad \text{and} \quad (q|_T)^*: H^*(B; \mathbb{F}_p) \rightarrow H^*(T; \mathbb{F}_p).$$

**3.3. Two lemmas.** In this section we recall two facts: an intersection lemma from [5, Lem. 4.3] and the computation of a particular Euler class from [10, Lem. 8].

**Lemma 3.2** (The intersection lemma). *Let*

- $k \geq 1$  be an integer, and  $p$  a prime,
- $B$  be an  $\mathbb{F}_p$ -orientable compact  $m$ -manifold,
- $q: E \rightarrow B$  be an  $n$ -dimensional real vector bundle whose mod- $p$  Euler class  $e \in H^n(B; \mathbb{F}_p)$  satisfies  $e^k \neq 0$ , and
- $T_0, \dots, T_k \subseteq E$  be compact subsets with the property that the induced maps

$$(q|_{T_i})^*: H^m(B; \mathbb{F}_p) \rightarrow H^m(T_i; \mathbb{F}_p),$$

for all  $0 \leq i \leq k$ , are injective.

Then

$$T_0 \cap \cdots \cap T_k \neq \emptyset.$$

Let  $G_n(\mathbb{R}^d)$  denote the Grassmann manifold of all  $n$ -dimensional subspaces in  $\mathbb{R}^d$ , and let  $\gamma^n(\mathbb{R}^d)$  be the corresponding canonical vector bundle over  $G_n(\mathbb{R}^d)$ . Furthermore, let  $\tilde{G}_n(\mathbb{R}^d)$  denote the oriented Grassmann manifold of all  $n$ -dimensional oriented subspaces in  $\mathbb{R}^d$ , and let  $\tilde{\gamma}^n(\mathbb{R}^d)$  be the corresponding canonical vector bundle over  $\tilde{G}_n(\mathbb{R}^d)$ . Then the “forgetting orientation” map  $\tilde{G}_n(\mathbb{R}^d) \rightarrow G_n(\mathbb{R}^d)$  is a double cover, and it induces a vector bundle map  $\tilde{\gamma}^n(\mathbb{R}^d) \rightarrow \gamma^n(\mathbb{R}^d)$  that is an isomorphism on fibers.

**Lemma 3.3** (Euler classes of the canonical bundles of real Grassmannians). *Let  $d$  and  $m$  be positive integers with  $0 \leq m \leq d-1$ , and let  $p$  be a prime.*

- (1) *If  $p = 2$  and  $\gamma := \gamma^{d-m}(\mathbb{R}^d)$ , then the  $m$ -th power of the Euler class of  $\gamma$  does not vanish, that is*

$$0 \neq e(\gamma)^m = w_{d-m}(\gamma)^m \in H^{(d-m)m}(G_{d-m}(\mathbb{R}^d); \mathbb{F}_2).$$

- (2) *If  $p$  is an odd prime,  $d-m$  is even, and  $\tilde{\gamma} := \tilde{\gamma}^{d-m}(\mathbb{R}^d)$ , then the  $m$ -th power of the mod- $p$  Euler class of  $\tilde{\gamma}$  does not vanish, that is*

$$0 \neq e(\tilde{\gamma})^m \in H^{(d-m)m}(\tilde{G}_{d-m}(\mathbb{R}^d); \mathbb{F}_p).$$

- (3) *If  $p$  is an odd prime,  $d-m$  is even, and  $\gamma := \gamma^{d-m}(\mathbb{R}^d)$ , then the  $m$ -th power of the mod- $p$  Euler class of  $\gamma$  does not vanish, that is*

$$0 \neq e(\gamma)^m \in H^{(d-m)m}(G_{d-m}(\mathbb{R}^d); \mathbb{F}_p).$$

The third part of the lemma is a consequence of the second part and the naturality property of the Euler class. The case  $p = 2$  of this lemma was proved by Dol’nikov in [7, Lem. p.112]. For  $p$  an odd prime,  $d \geq 3$  an odd integer, and  $d-m$  even the lemma was first proved by Živaljević [19, Prop.4.9].

#### 4. PROOFS

Now, combining the methods presented in Section 1, Theorem 3.1 and Lemmas 3.2 and 3.3, we prove our main results, Theorems 2.1 and 2.2.

**4.1. Proof of the topological generalized transversal Van Kampen–Flores theorem.** Let  $B := G_{d-m}(\mathbb{R}^d)$  be the Grassmann manifold, and let  $\gamma := \gamma^{d-m}(\mathbb{R}^d)$  be the canonical bundle. Without loss of generality we can assume that  $m \geq 1$ . The proof of Theorem 2.1 is done in several steps.

4.1.1. Fix an integer  $0 \leq \ell \leq m$ , and define  $K_\ell := (\Delta_{N_\ell})_{\Delta(2)}^{\times r_\ell}$  to be the  $r_\ell$ -fold 2-wise deleted product of the simplex  $\Delta_{N_\ell}$ . According to [2, Lem. 1] the complex  $K_\ell$  is an  $(N_\ell - r_\ell + 1)$ -dimensional and  $(N_\ell - r_\ell)$ -connected CW complex. The symmetric group  $\mathfrak{S}_{r_\ell}$  acts freely on  $K_\ell$  by permuting factors in the product, that is  $\pi \cdot (x_1, \dots, x_{r_\ell}) := (x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(r_\ell)})$ , for  $\pi \in \mathfrak{S}_{r_\ell}$  and  $(x_1, \dots, x_{r_\ell}) \in K_\ell$ .

Consider the regular embedding  $\text{reg} : (\mathbb{Z}/p)^{e_\ell} \rightarrow \mathfrak{S}_{r_\ell}$  of the elementary abelian group  $(\mathbb{Z}/p)^{e_\ell}$ , as explained in [1, Ex.2.7, p.100]. It is given by the left translation action of  $(\mathbb{Z}/p)^{e_\ell}$  on itself. To every element  $g \in (\mathbb{Z}/p)^{e_\ell}$  we associate the permutation  $L_g : (\mathbb{Z}/p)^{e_\ell} \rightarrow (\mathbb{Z}/p)^{e_\ell}$  from  $\text{Sym}((\mathbb{Z}/p)^{e_\ell}) \cong \mathfrak{S}_{r_\ell}$  given by  $L_g(x) = g + x$ . Thus, the elementary abelian group  $G_\ell := (\mathbb{Z}/p)^{e_\ell}$  is identified with subgroup  $\text{im}(\text{reg})$  of the symmetric group  $\mathfrak{S}_{r_\ell}$ . Consequently,  $K_\ell$  is a free  $G_\ell$ -space.

Furthermore, let  $\mathbb{R}^{r_\ell}$  be a vector space with the (left) action of the symmetric group  $\mathfrak{S}_{r_\ell}$  given by permutation of coordinates. Then the subspace  $W_{r_\ell} := \{(t_1, \dots, t_{r_\ell}) \in \mathbb{R}^{r_\ell} : \sum t_i = 0\}$  is a  $\mathfrak{S}_{r_\ell}$ -invariant subspace. The group  $G_\ell$  acts on both  $\mathbb{R}^{r_\ell}$  and  $W_{r_\ell}$  via the regular embedding.

Let  $\tau_\ell$  be the trivial vector bundle  $B \times W_{r_\ell} \rightarrow B$ . The action of  $G_\ell$  on  $W_{r_\ell}$  makes  $\tau_\ell$  into a  $G_\ell$ -equivariant vector bundle. Next,  $\gamma^{\oplus r_\ell}$  is also a  $G_\ell$ -equivariant vector bundle where the action is given by permutation of summands in the Whitney sum. Then the vector bundle

$$\xi_\ell := \tau_\ell \oplus \gamma^{\oplus r_\ell} \tag{1}$$

inherits the structure of a  $G_\ell$ -equivariant vector bundle via the diagonal action. Let  $E(\cdot)$  denote the total space of a vector bundle. Since the  $G_\ell$  fixed point set of  $W_{r_\ell}$  is just zero, that is  $W_{r_\ell}^{G_\ell} = \{0\}$ , the fixed point set of the total space of  $\xi_\ell$  is

$$E(\xi_\ell)^{G_\ell} = E(\tau_\ell \oplus \gamma^{\oplus r_\ell})^{G_\ell} \cong E(\gamma^{\oplus r_\ell})^{G_\ell} \cong E(\gamma).$$

4.1.2. We define a continuous  $G_\ell$ -equivariant bundle map  $\Phi_\ell: B \times K_\ell \rightarrow E(\xi_\ell)$  as follows: For the point  $(b, (x_1, \dots, x_{r_\ell})) \in B \times K_\ell$  let

$$\begin{aligned} \Phi_\ell(b, (x_1, \dots, x_{r_\ell})) := & \\ & \left( b, (\text{dist}(x_1, \text{sk}_{k_\ell}(\Delta_{N_\ell})) - a(x_1, \dots, x_{r_\ell}), \dots, \text{dist}(x_{r_\ell}, \text{sk}_{k_\ell}(\Delta_{N_\ell})) - a(x_1, \dots, x_{r_\ell})) \right) \oplus \\ & ((q_b \circ f_\ell)(x_1) \oplus \dots \oplus (q_b \circ f_\ell)(x_{r_\ell})), \end{aligned}$$

where

- $q: \mathbb{R}^d \rightarrow b$  is the orthogonal projection onto the  $(d - m)$ -dimensional subspace  $b \in B$  of  $\mathbb{R}^d$ ,
- $\text{dist}(\cdot, \text{sk}_{k_\ell}(\Delta_{N_\ell}))$  denotes the distance function to the  $k_\ell$ -skeleton of the simplex  $\Delta_{N_\ell}$ , and
- $a(x_1, \dots, x_{r_\ell}) = \frac{1}{r_\ell} (\text{dist}(x_1, \text{sk}_{k_\ell}(\Delta_{N_\ell})) + \dots + \text{dist}(x_{r_\ell}, \text{sk}_{k_\ell}(\Delta_{N_\ell})))$ .

Next we consider the compact subsets

$$S_\ell := \Phi_\ell^{-1}(E(\xi_\ell)^{G_\ell}) \quad \text{and} \quad T_\ell := \Phi_\ell(S_\ell) = \text{im}(\Phi_\ell) \cap E(\xi_\ell)^{G_\ell}.$$

where  $T_\ell \subseteq E(\xi_\ell)^{G_\ell} \cong E(\gamma)$ . The set  $S_\ell$  contains of all points  $(b, (x_1, \dots, x_{r_\ell})) \in B \times K_\ell$  such that

$$\text{dist}(x_1, \text{sk}_{k_\ell}(\Delta_{N_\ell})) = \dots = \text{dist}(x_{r_\ell}, \text{sk}_{k_\ell}(\Delta_{N_\ell})) \quad \text{and} \quad (q_b \circ f_\ell)(x_1) = \dots = (q_b \circ f_\ell)(x_{r_\ell}).$$

Since  $(x_1, \dots, x_{r_\ell}) \in K_\ell$ , then there exist  $r_\ell$  pairwise disjoint faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  of  $\Delta_{N_\ell}$  such that

$$(x_1, \dots, x_{r_\ell}) \in \text{relint } \sigma_1^\ell \times \dots \times \text{relint } \sigma_{r_\ell}^\ell,$$

and at least one of the faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  belongs to the  $k_\ell$ -skeleton of the simplex  $\Delta_{N_\ell}$  [4, Lem. 4.2 (iii)]. Indeed, if this would not be true all the faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  would be at least  $(k_\ell + 1)$ -dimensional, implying the following contradiction

$$N_\ell + 1 = |\Delta_{N_\ell}| \geq |\sigma_1^\ell| + \dots + |\sigma_{r_\ell}^\ell| \geq r_\ell(k_\ell + 2) \geq r_\ell(\lceil \frac{r_\ell - 1}{r_\ell} d \rceil + 2) \geq (r_\ell - 1)(d + 2) + 2 = N_\ell + 2.$$

Therefore, at least one of the faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  lies in  $\text{sk}_{k_\ell}(\Delta_{N_\ell})$  and consequently

$$\text{dist}(x_1, \text{sk}_{k_\ell}(\Delta_{N_\ell})) = \dots = \text{dist}(x_{r_\ell}, \text{sk}_{k_\ell}(\Delta_{N_\ell})) = 0,$$

implying that all the faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  lies in  $\text{sk}_{k_\ell}(\Delta_{N_\ell})$ .

Thus, in order to conclude the proof of Theorem 2.1 we need to show that

$$\emptyset \neq T_0 \cap \dots \cap T_m \subseteq E(\gamma),$$

and for that we would like to use Lemma 3.2.

4.1.3. First, let  $0 \leq \ell \leq m$  and let  $e_\ell = 0$ . Then  $r_\ell = 1$ ,  $N_\ell = 0$ ,  $K_\ell = \Delta_{N_\ell}$  is a point,  $G_\ell$  is the trivial group, and  $S_\ell = B \times K_\ell$ . Consider the commutative diagram induced by the bundle map  $\Phi_\ell: B \times K_\ell \rightarrow E(\xi_\ell)$  and the corresponding diagram in cohomology:

$$\begin{array}{ccccc} B \times K_\ell = S_\ell & \xrightarrow{\Phi_\ell|_{S_\ell}} & T_\ell & & H^*(B; \mathbb{F}_p) \cong H^*(S_\ell; \mathbb{F}_p) \xleftarrow{(\Phi_\ell|_{S_\ell})^*} H^*(T_\ell; \mathbb{F}_p) \\ & \searrow p_1 & \swarrow q_\ell|_{T_\ell} & & \swarrow p_1^* & \searrow (q_\ell|_{T_\ell})^* \\ & & B & & H^*(B; \mathbb{F}_p). \end{array}$$

Since  $K_\ell$  is a point the map  $p_1^*$  induced by the projection  $p_1$  is the identity map. Consequently, the map

$$(q_\ell|_{T_\ell})^*: H^*(B; \mathbb{F}_p) \rightarrow H^*(T_\ell; \mathbb{F}_p).$$

is an injection.

4.1.4. Next, let  $0 \leq \ell \leq m$ , and let  $e_\ell > 0$ . Now we apply Theorem 3.1 to the  $G_\ell$ -equivariant bundle map  $\Phi_\ell: B \times K_\ell \rightarrow E(\xi_\ell)$ . In order to do so we check the necessary assumptions. Since

- $G_\ell = (\mathbb{Z}_p)^{e_\ell}$  is an elementary abelian group,
- $B = G_{d-m}(\mathbb{R}^d)$  is a connected space with the trivial  $G_\ell$ -action,
- $q_\ell: E(\xi_\ell) \rightarrow B$  is a  $G_\ell$ -equivariant vector bundle where all fibers carry the same  $G_\ell$ -representation,
- $q_\ell|_{E(\xi_\ell)^{G_\ell}}: E(\xi_\ell)^{G_\ell} \rightarrow B$  is the fixed-point subbundle with the  $G_\ell$ -invariant orthogonal complement subbundle  $q_\ell|_{C_\ell}: C_\ell \rightarrow B$ ,  $(E(\xi_\ell) = C_\ell \oplus E(\xi_\ell)^{G_\ell})$ ,
- $F_\ell$  is the fiber of the vector bundle  $q_\ell|_{C_\ell}: C_\ell \rightarrow B$  over the point  $b \in B$ ,
- $\pi_1(B)$  acts trivially on the cohomology of the sphere  $H^*(S(F_\ell); \mathbb{F}_p)$ ,



- the Euler class  $0 \neq \alpha_\ell \in H^{(r_\ell-1)(d-m+1)}(G_\ell; \mathbb{F}_p)$  of the vector bundle  $F_\ell \rightarrow EG_\ell \times_{G_\ell} F_\ell \rightarrow BG_\ell$  does not vanish, more precisely

$$\alpha_\ell = \left( \prod_{(a_1, \dots, a_{e_\ell}) \in \mathbb{F}_p^{e_\ell} \setminus \{0\}} (a_1 t_1 + \dots + a_{e_\ell} t_{e_\ell}) \right)^{\frac{d-m+1}{2}},$$

- $\text{Index}_{G_\ell}^{\text{pt}}(K_\ell; \mathbb{F}_p) \subseteq H^{\geq (r_\ell-1)(d-m+1)+1}(G_\ell; \mathbb{F}_p)$  because  $K_\ell$  is  $((r_\ell-1)(d-m+1)-1)$ -connected, we have that  $\alpha_\ell \notin \text{Index}_{G_\ell}^{\text{pt}}(K_\ell; \mathbb{F}_p)$  and Theorem 3.1 can be applied on the  $G_\ell$ -equivariant bundle map  $\Phi_\ell: B \times K_\ell \rightarrow E(\xi_\ell)$ . Thus, the following map in Čech cohomology induced by  $q_\ell$  is injective:

$$(q_\ell|_{T_\ell})^*: H^*(B; \mathbb{F}_p) \rightarrow H^*(T_\ell; \mathbb{F}_p).$$

4.1.5. Finally, Lemma 3.2 comes into play. Since,

- $T_\ell$  is a compact subset of  $E(\gamma)$  for every  $0 \leq \ell \leq m$ ,
  - $(q_\ell|_{T_\ell})^*: H^*(B; \mathbb{F}_p) \rightarrow H^*(T_\ell; \mathbb{F}_p)$  is injective for every  $0 \leq \ell \leq m$ , and
  - $0 \neq e(\gamma)^m \in H^{(d-m)m}(B; \mathbb{F}_p)$  according to  $p(d-m)$  being even and Lemma 3.3,
- we can apply Lemma 3.2 and get that

$$T_0 \cap \dots \cap T_m \neq \emptyset.$$

This concludes the proof of Theorem 2.1.  $\square$

**4.2. Proof of the topological transversal weak colored Tverberg theorem.** Let  $B := G_{d-m}(\mathbb{R}^d)$  be the Grassmann manifold, and let  $\gamma := \gamma^{d-m}(\mathbb{R}^d)$  be the canonical bundle. Without loss of generality we can assume that  $m \geq 1$ . The proof of Theorem 2.2 is done in the footsteps of the proof of Theorem 2.1. The only difference will occur in the definition of the bundles  $\tau_\ell$  and consequently bundle maps  $\Phi_\ell$ .

4.2.1. Again, fix an integer  $0 \leq \ell \leq m$ , and define  $K_\ell := (\Delta_{N_\ell})_{\Delta(2)}^{\times r_\ell}$ . As we have seen the complex  $K_\ell$  is an  $(N_\ell - r_\ell + 1)$ -dimensional and  $(N_\ell - r_\ell)$ -connected CW complex. The symmetric group  $\mathfrak{S}_{r_\ell}$  acts freely on  $K_\ell$  by permuting factors in the product.

The regular embedding  $\text{reg}: (\mathbb{Z}/p)^{e_\ell} \rightarrow \mathfrak{S}_{r_\ell}$  of the elementary abelian group  $(\mathbb{Z}/p)^{e_\ell}$  identifies elementary abelian group  $G_\ell := (\mathbb{Z}/p)^{e_\ell}$  with a subgroup  $\text{im}(\text{reg})$  of the symmetric group  $\mathfrak{S}_{r_\ell}$ .

Once more,  $\mathbb{R}^{r_\ell}$  is a vectors space with the (left) action of the symmetric group  $\mathfrak{S}_{r_\ell}$  given by permutation of coordinates. The subspace  $W_{r_\ell} := \{(t_1, \dots, t_{r_\ell}) \in \mathbb{R}^{r_\ell} : \sum t_i = 0\}$  is a  $\mathfrak{S}_{r_\ell}$ -invariant subspace, and  $G_\ell$  acts on both  $\mathbb{R}^{r_\ell}$  and  $W_{r_\ell}$  via the regular embedding.

Let  $\tau_\ell$  be the trivial vector bundle  $B \times W_{r_\ell}^{\oplus d+1} \rightarrow B$ . The action of  $G_\ell$  on  $W_{r_\ell}^{\oplus d+1}$  is diagonal and makes  $\tau_\ell$  into a  $G_\ell$ -equivariant vector bundle. As we have seen,  $\gamma^{\oplus r_\ell}$  is also a  $G_\ell$ -equivariant vector bundle. Thus the vector bundle

$$\xi_\ell := \tau_\ell \oplus \gamma^{\oplus r_\ell} \tag{2}$$

inherits the structure of a  $G_\ell$ -equivariant vector bundle via the diagonal action. Since  $(W_{r_\ell}^{\oplus d+1})^{G_\ell} = \{0\}$ , the fixed point set of the total space of  $\xi_\ell$  is

$$E(\xi_\ell)^{G_\ell} = E(\tau_\ell \oplus \gamma^{\oplus r_\ell})^{G_\ell} \cong E(\gamma^{\oplus r_\ell})^{G_\ell} \cong E(\gamma).$$

4.2.2. The vertices of the simplex  $\Delta_{N_\ell}$  are colored by  $d+1$  colors, where each color class has cardinality at most  $2r_\ell - 1$ . Set  $\text{vert}(\Delta_N) = C_0 \sqcup \dots \sqcup C_d$  where  $|C_i| \leq 2r_\ell - 1$  for all  $0 \leq i \leq d$ . Following the idea from [4, Lem. 4.2 (ii)] we define  $\Sigma_i^\ell$ ,  $0 \leq i \leq d$ , to be the subcomplex of  $\Delta_{N_\ell}$  consisting of all faces with at most one vertex in  $C_i$ . Then the rainbow subcomplex  $R_{N_\ell}$  coincides with the intersection  $\Sigma_0^\ell \cap \dots \cap \Sigma_d^\ell$ .

Now we define a continuous  $G_\ell$ -equivariant bundle map  $\Phi_\ell: B \times K_\ell \rightarrow E(\xi_\ell)$  as follows: For the point  $(b, (x_1, \dots, x_{r_\ell})) \in B \times K_\ell$  let

$$\begin{aligned} \Phi_\ell(b, (x_1, \dots, x_{r_\ell})) &:= (b, \text{dist}(x_1, \Sigma_0^\ell) - a_0(x_1, \dots, x_{r_\ell}), \dots, \text{dist}(x_{r_\ell}, \Sigma_0^\ell) - a_0(x_1, \dots, x_{r_\ell})) \oplus \\ &\quad \dots \\ &\quad (b, \text{dist}(x_1, \Sigma_d^\ell) - a_d(x_1, \dots, x_{r_\ell}), \dots, \text{dist}(x_{r_\ell}, \Sigma_d^\ell) - a_d(x_1, \dots, x_{r_\ell})) \oplus \\ &\quad ((q_b \circ f_\ell)(x_1) \oplus \dots \oplus (q_b \circ f_\ell)(x_{r_\ell})) \end{aligned}$$

where

- $q: \mathbb{R}^d \rightarrow b$  is the orthogonal projection onto the  $(d-m)$ -dimensional subspace  $b \in B$  of  $\mathbb{R}^d$ ,
- $\text{dist}(\cdot, \Sigma_i^\ell)$  denotes the distance function to the subcomplex  $\Sigma_i^\ell$  where  $0 \leq i \leq d$ , and
- $a_i(x_1, \dots, x_{r_\ell}) = \frac{1}{r_\ell} (\text{dist}(x_1, \Sigma_i^\ell) + \dots + \text{dist}(x_{r_\ell}, \Sigma_i^\ell))$  for  $0 \leq i \leq d$ .

Again, we consider the compact subsets

$$S_\ell := \Phi_\ell^{-1}(E(\xi_\ell)^{G_\ell}) \quad \text{and} \quad T_\ell := \Phi_\ell(S_\ell) = \text{im}(\Phi_\ell) \cap E(\xi_\ell)^{G_\ell}.$$

where  $T_\ell \subseteq E(\xi_\ell)^{G_\ell} \cong E(\gamma)$ . The set  $S_\ell$  contains of all points  $(b, (x_1, \dots, x_{r_\ell})) \in B \times K_\ell$  such that

$$\text{dist}(x_1, \Sigma_0^\ell) = \dots = \text{dist}(x_{r_\ell}, \Sigma_0^\ell), \quad \dots, \quad \text{dist}(x_1, \Sigma_d^\ell) = \dots = \text{dist}(x_{r_\ell}, \Sigma_d^\ell),$$

and

$$(q_b \circ f_\ell)(x_1) = \dots = (q_b \circ f_\ell)(x_{r_\ell}).$$

Since the point  $(x_1, \dots, x_{r_\ell}) \in K_\ell$ , then we can find  $r_\ell$  unique pairwise disjoint faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  with the property that

$$(x_1, \dots, x_{r_\ell}) \in \text{relint } \sigma_1^\ell \times \dots \times \text{relint } \sigma_{r_\ell}^\ell.$$

Moreover, for every  $i$  in the range  $0 \leq i \leq d$  there exists at least one of the faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  that belongs to the subcomplex  $\Sigma_i$  of the simplex  $\Delta_{N_\ell}$ , [4, Lem. 4.2 (ii)]. Indeed, if this would not be true for an index  $i$  then all the faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  would have at least two vertices from the set  $C_i$  and we have the contradiction

$$2r_\ell - 1 \geq |C_i| \geq |\sigma_1^\ell \cap C_i| + \dots + |\sigma_{r_\ell}^\ell \cap C_i| \geq 2r_\ell.$$

Thus for every index  $i$  at least one of the faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  lies in  $\Sigma_i$  and consequently

$$\text{dist}(x_1, \Sigma_0^\ell) = \dots = \text{dist}(x_{r_\ell}, \Sigma_0^\ell) = 0, \quad \dots, \quad \text{dist}(x_1, \Sigma_d^\ell) = \dots = \text{dist}(x_{r_\ell}, \Sigma_d^\ell) = 0,$$

implying that all the faces  $\sigma_1^\ell, \dots, \sigma_{r_\ell}^\ell$  lie in every subcomplex  $\Sigma_i$ , meaning they belong to the intersection  $\Sigma_0^\ell \cap \dots \cap \Sigma_d^\ell$  – the rainbow subcomplex.

Therefore, in order to finalize the proof of Theorem 2.2 we need to show, as in the proof of the previous theorem, that

$$\emptyset \neq T_0 \cap \dots \cap T_m \subseteq E(\gamma).$$

4.2.3. Again first assume that  $0 \leq \ell \leq m$  and that  $e_\ell = 0$ . We proceed as in Section 4.1.3. Now  $r_\ell = 1$ ,  $N_\ell = 0$ ,  $K_\ell = \Delta_{N_\ell}$  is a point,  $G_\ell$  is the trivial group, and  $S_\ell = B \times K_\ell$ . We consider the commutative diagrams induced by the bundle map  $\Phi_\ell: B \times K_\ell \rightarrow E(\xi_\ell)$ :

$$\begin{array}{ccc} B \times K_\ell = S_\ell & \xrightarrow{\Phi_\ell|_{S_\ell}} & T_\ell \\ & \searrow p_1 & \swarrow q_\ell|_{T_\ell} \\ & B & \end{array} \quad \begin{array}{ccc} H^*(B; \mathbb{F}_p) \cong H^*(S_\ell; \mathbb{F}_p) & \xleftarrow{(\Phi_\ell|_{S_\ell})^*} & H^*(T_\ell; \mathbb{F}_p) \\ & \nwarrow p_1^* & \nearrow (q_\ell|_{T_\ell})^* \\ & H^*(B; \mathbb{F}_p). & \end{array}$$

The map  $p_1^*$  induced by the projection  $p_1$  is the identity map. Hence, the map in cohomology

$$(q_\ell|_{T_\ell})^*: H^*(B; \mathbb{F}_p) \rightarrow H^*(T_\ell; \mathbb{F}_p).$$

is an injection.

4.2.4. Let  $0 \leq \ell \leq m$  and  $e_\ell > 0$ . We apply Theorem 3.1 to the  $G_\ell$ -equivariant bundle map  $\Phi_\ell: B \times K_\ell \rightarrow E(\xi_\ell)$ . For that we check the necessary assumptions. Since

- $G_\ell = (\mathbb{Z}_p)^{e_\ell}$  is an elementary abelian group,
- $B = G_{d-m}(\mathbb{R}^d)$  is a connected space with the trivial  $G_\ell$ -action,
- $q_\ell: E(\xi_\ell) \rightarrow B$  is a  $G_\ell$ -equivariant vector bundle where all fibers carry the same  $G_\ell$ -representation,
- $q_\ell|_{E(\xi_\ell)^{G_\ell}}: E(\xi_\ell)^{G_\ell} \rightarrow B$  is the fixed-point subbundle with the  $G_\ell$ -invariant orthogonal complement subbundle  $q_\ell|_{C_\ell}: C_\ell \rightarrow B$ ,  $(E(\xi_\ell) = C_\ell \oplus E(\xi_\ell)^{G_\ell})$ ,
- $F_\ell$  is the fiber of the vector bundle  $q_\ell|_{C_\ell}: C_\ell \rightarrow B$  over the point  $b \in B$ ,
- $\pi_1(B)$  acts trivially on the cohomology of the sphere  $H^*(S(F_\ell); \mathbb{F}_p)$ ,
- the Euler class  $0 \neq \alpha_\ell \in H^{(r_\ell-1)(2d-m+1)}(G_\ell; \mathbb{F}_p)$  of the vector bundle  $F_\ell \rightarrow \text{EG}_\ell \times_{G_\ell} F_\ell \rightarrow \text{BG}_\ell$  does not vanish, more precisely

$$\alpha_\ell = \left( \prod_{(a_1, \dots, a_{e_\ell}) \in \mathbb{F}_p^{e_\ell} \setminus \{0\}} (a_1 t_1 + \dots + a_{e_\ell} t_{e_\ell}) \right)^{\frac{2d-m+1}{2}},$$

◦  $\text{Index}_{G_\ell}^{\text{pt}}(K_\ell; \mathbb{F}_p) \subseteq H^{\geq (r_\ell-1)(2d-m+1)+1}(G_\ell; \mathbb{F}_p)$  because  $K_\ell$  is  $((r_\ell-1)(2d-m+1)-1)$ -connected, we conclude that  $\alpha_\ell \notin \text{Index}_{G_\ell}^{\text{pt}}(K_\ell; \mathbb{F}_p)$ , and therefore Theorem 3.1 can be applied on the  $G_\ell$ -equivariant bundle map  $\Phi_\ell: B \times K_\ell \rightarrow E(\xi_\ell)$ . Consequently, the following map induced by  $q_\ell$  is injective:

$$(q_\ell|_{T_\ell})^*: H^*(B; \mathbb{F}_p) \rightarrow H^*(T_\ell; \mathbb{F}_p).$$



4.2.5. In the final step we apply Lemma 3.2. Since,

- $T_\ell$  is a compact subset of  $E(\gamma)$  for every  $0 \leq \ell \leq m$ ,
  - $(q_\ell|_{T_\ell})^*: H^*(B; \mathbb{F}_p) \longrightarrow H^*(T_\ell; \mathbb{F}_p)$  is injective for every  $0 \leq \ell \leq m$ , and
  - $0 \neq e(\gamma)^m \in H^{(d-m)m}(B; \mathbb{F}_p)$  does not vanish according to  $p(d-m)$  being even and Lemma 3.3,
- we can apply Lemma 3.2 and get that

$$T_0 \cap \cdots \cap T_m \neq \emptyset.$$

This concludes the proof of Theorem 2.2. □

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INST. MATH., FU BERLIN, ARNIMALLEE 2, 14195 BERLIN, GERMANY  
 MAT. INSTITUT SANU, KNEZ MIHAILOVA 36, 11001 BEOGRAD, SERBIA  
*E-mail address:* [blagojevic@math.fu-berlin.de](mailto:blagojevic@math.fu-berlin.de)

MAT. INSTITUT SANU, KNEZ MIHAILOVA 36, 11001 BEOGRAD, SERBIA  
*E-mail address:* [aleksandra1973@gmail.com](mailto:aleksandra1973@gmail.com)

INST. MATH., FU BERLIN, ARNIMALLEE 2, 14195 BERLIN, GERMANY  
*E-mail address:* [ziegler@math.fu-berlin.de](mailto:ziegler@math.fu-berlin.de)